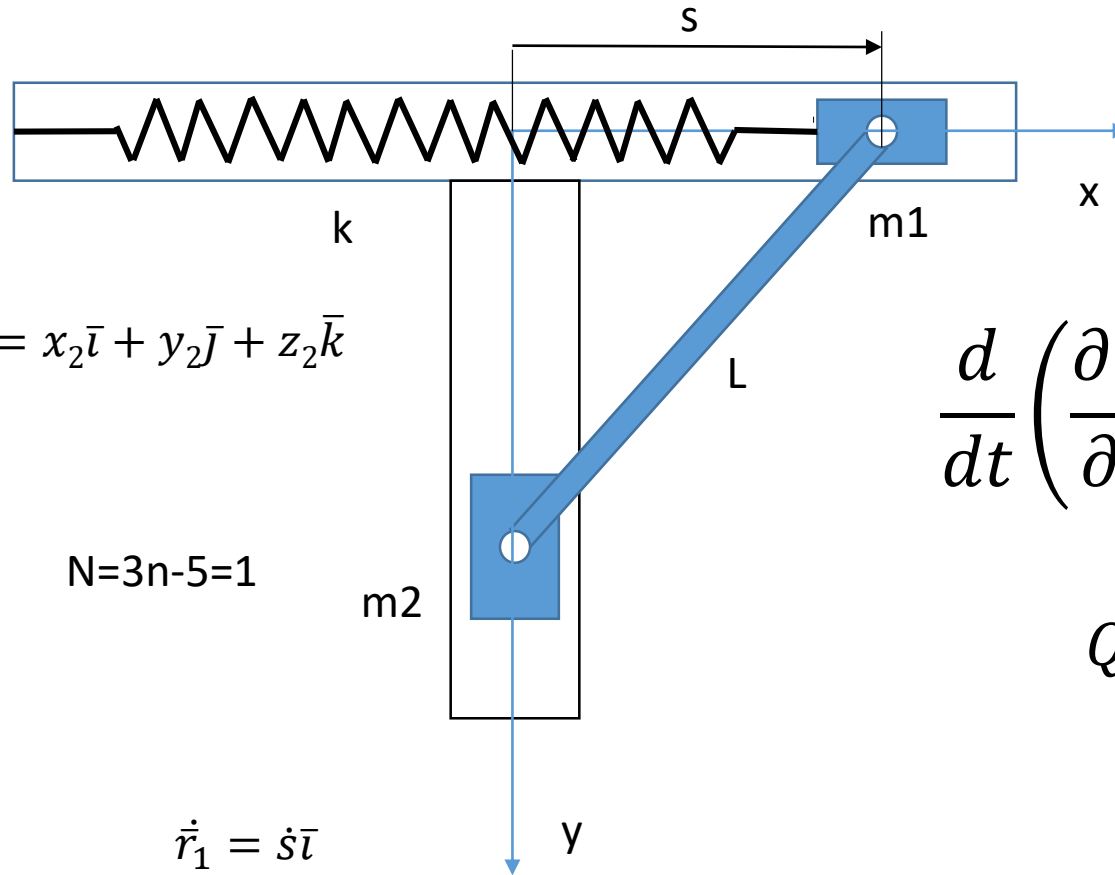


Przykład

Dane: m_1, m_2, L, k



$$\bar{r}_1 = x_1 \bar{i} + y_1 \bar{j} + z_1 \bar{k}, \quad \bar{r}_2 = x_2 \bar{i} + y_2 \bar{j} + z_2 \bar{k}$$

$n=2$

więzy

$$z_1=0, z_2=0, y_1=0, x_2=0$$

$$x_1^2 + y_2^2 = L^2$$

$$N=3n-5=1$$

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_m} \right) - \frac{\partial E_k}{\partial q_m} = Q_m$$

$$Q_m = \left[\sum_{i=1}^n \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_m} \right]$$

$$\bar{r}_1 = s \bar{i}$$

$$\dot{\bar{r}}_1 = \dot{s} \bar{i}$$

$$\bar{r}_2 = y_2 \bar{j} = \sqrt{L^2 - s^2} \bar{j}$$

$$\dot{\bar{r}}_2 = \frac{-2s\dot{s}}{2\sqrt{L^2 - s^2}} \bar{j}$$

$$E_k = \frac{1}{2} m_1 \dot{\bar{r}}_1^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2^2 = \frac{1}{2} m_1 \dot{s}^2 + \frac{1}{2} m_2 \frac{s^2 \dot{s}^2}{L^2 - s^2}$$

$$\frac{\partial E_k}{\partial \dot{s}} = m_1 \dot{s} + m_2 \frac{s^2 \dot{s}}{L^2 - s^2}$$

$$\frac{\partial E_k}{\partial s} = \frac{1}{2} m_2 \dot{s}^2 \left[\frac{2s(L^2 - s^2) - s^2(-2s)}{(L^2 - s^2)^2} \right] = m_2 \dot{s}^2 \frac{sL^2}{(L^2 - s^2)^2}$$

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{s}} \right) = m_1 \ddot{s} + m_2 \left[\frac{\frac{d}{dt} (s^2 \dot{s}) (L^2 - s^2) - (s^2 \dot{s}) (-2s \dot{s})}{(L^2 - s^2)^2} \right] = m_1 \ddot{s} + m_2 \left[\frac{(2s \dot{s}^2 + s^2 \ddot{s}) (L^2 - s^2) + 2s^3 \dot{s}^2}{(L^2 - s^2)^2} \right]$$

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{s}} \right) = m_1 \ddot{s} + m_2 \frac{s^2 \ddot{s}}{(L^2 - s^2)} + m_2 \left[\frac{2s \dot{s}^2 L^2}{(L^2 - s^2)^2} \right]$$

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{s}} \right) - \frac{\partial E_k}{\partial s} = m_1 \ddot{s} + m_2 \frac{s^2 \ddot{s}}{(L^2 - s^2)} + m_2 \dot{s}^2 \frac{2s L^2}{(L^2 - s^2)^2} - m_2 \dot{s}^2 \frac{s L^2}{(L^2 - s^2)^2} = m_1 \ddot{s} + m_2 \frac{s^2 \ddot{s}}{(L^2 - s^2)} + m_2 \frac{\dot{s}^2 s L^2}{(L^2 - s^2)^2}$$

$$Q = \left[\sum_{i=1}^n \bar{F}_i \frac{\partial \bar{r}_i}{\partial s} \right] = \bar{F}_1 \frac{\partial \bar{r}_1}{\partial s} + \bar{F}_2 \frac{\partial \bar{r}_2}{\partial s} \quad \bar{F}_1 = -ks\bar{i} + m_1 g \bar{j} \quad \bar{F}_2 = m_2 g \bar{j}$$

$$\frac{\partial \bar{r}_1}{\partial s} = \bar{i} \quad \frac{\partial \bar{r}_2}{\partial s} = \frac{-2s}{2\sqrt{L^2 - s^2}} \bar{j} = \frac{-s}{\sqrt{L^2 - s^2}} \bar{j}$$

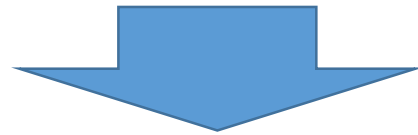
$$Q = \bar{i}(-ks\bar{i} + m_1 g \bar{j}) + \frac{-s}{\sqrt{L^2 - s^2}} \bar{j} \cdot m_2 g \bar{j} = -ks - \frac{m_2 g s}{\sqrt{L^2 - s^2}}$$

$$m_1 \ddot{s} + m_2 \frac{s^2 \ddot{s}}{(L^2 - s^2)} + m_2 \frac{\dot{s}^2 s L^2}{(L^2 - s^2)^2} + ks + \frac{m_2 g s}{\sqrt{L^2 - s^2}} = 0$$

Równanie jest nieliniowe i skomplikowane

Pierwsze uproszczenie to: małe drgania $\longrightarrow s \ll L \longrightarrow L^2 - s^2 = L^2$

Elementy wyższego rzędu $s^2 \ddot{s}$ i $\dot{s}^2 s$ zakładamy, że są równe zero



$$m_1 \ddot{s} + s \left(k + \frac{m_2 g}{L} \right) = 0$$

$$\omega_w = \sqrt{\frac{k + \frac{m_2 g}{L}}{m_1}}$$

oscylator harmoniczny

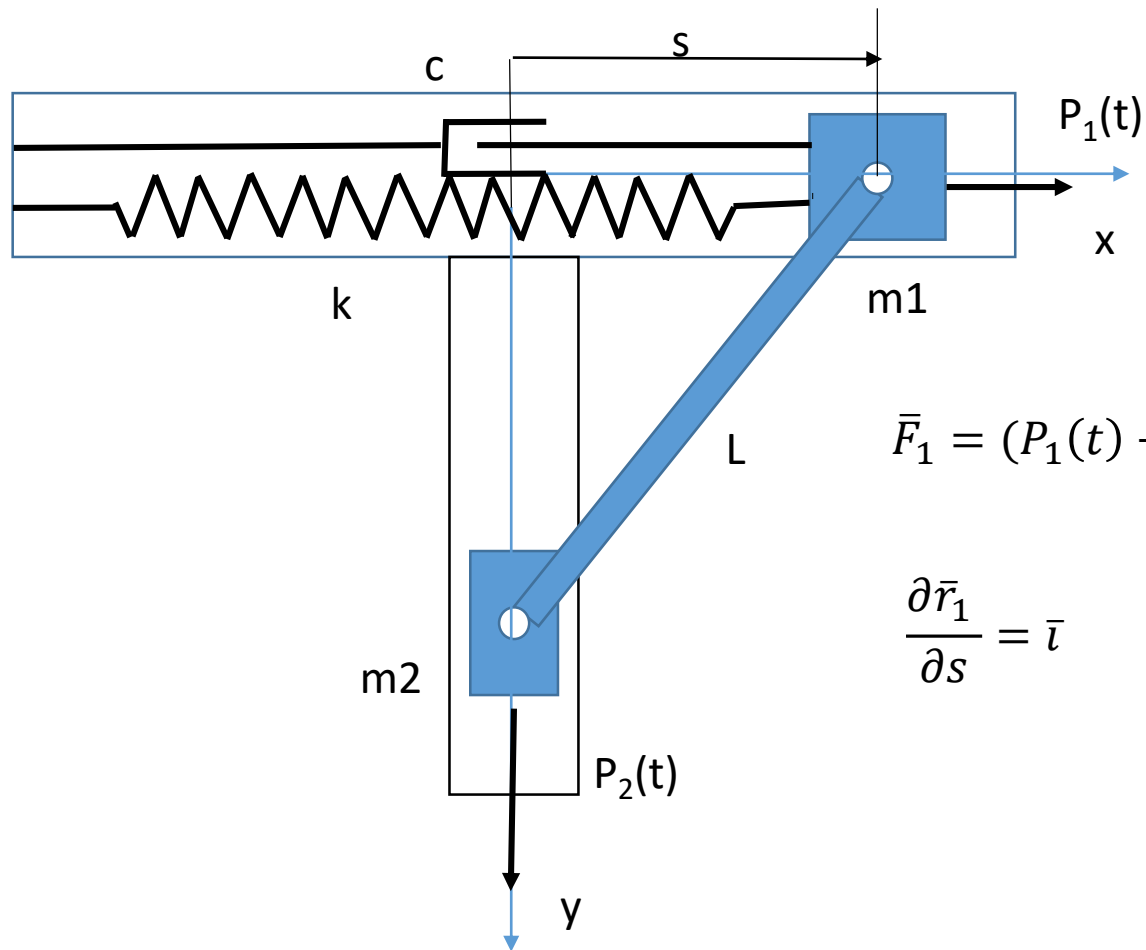
$$\ddot{x} + \omega_w^2 x = 0$$

ω_w - częstość drgań własnych

$$\omega_w = 2\pi f_w \longleftarrow \text{częstotliwość}$$

$$T = \frac{1}{f_w} \text{ okres drgań}$$

teraz dodajmy jeszcze tłumienie i siły wymuszające



$$Q = \left[\sum_{i=1}^n \bar{F}_i \frac{\partial \bar{r}_i}{\partial s} \right] = \bar{F}_1 \frac{\partial \bar{r}_1}{\partial s} + \bar{F}_2 \frac{\partial \bar{r}_2}{\partial s}$$

$$\bar{F}_1 = (P_1(t) - ks - c\dot{s})\bar{i} + m_1 g \bar{j} \quad \bar{F}_2 = (P_2(t) + m_2 g)\bar{j}$$

$$\frac{\partial \bar{r}_1}{\partial s} = \bar{i}$$

$$\frac{\partial \bar{r}_2}{\partial s} = \frac{-s}{\sqrt{L^2 - s^2}} \bar{j}$$

$$Q = \bar{i}((P_1(t) - ks - c\dot{s})\bar{i} + m_1 g \bar{j}) + \frac{-s}{\sqrt{L^2 - s^2}} \bar{j} \cdot (P_2(t) + m_2 g)\bar{j} = P_1(t) - ks - c\dot{s} - \frac{s}{\sqrt{L^2 - s^2}} (P_2(t) + m_2 g)$$

więc równanie będzie miało postać:

$$m_1 \ddot{s} + ks + c\dot{s} + m_2 \frac{s^2 \ddot{s}}{(L^2 - s^2)} + m_2 \frac{\dot{s}^2 s L^2}{(L^2 - s^2)^2} + \frac{m_2 g s}{\sqrt{L^2 - s^2}} = P_1(t) - \frac{P_2(t)s}{\sqrt{L^2 - s^2}}$$

małe drgania



$$s \ll L$$

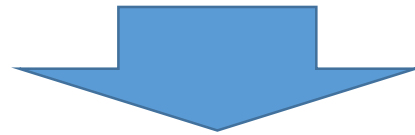


$$L^2 - s^2 = L^2$$

Elementy wyższego rzędu

$$s^2 \ddot{s} \quad \text{i} \quad \dot{s}^2 s$$

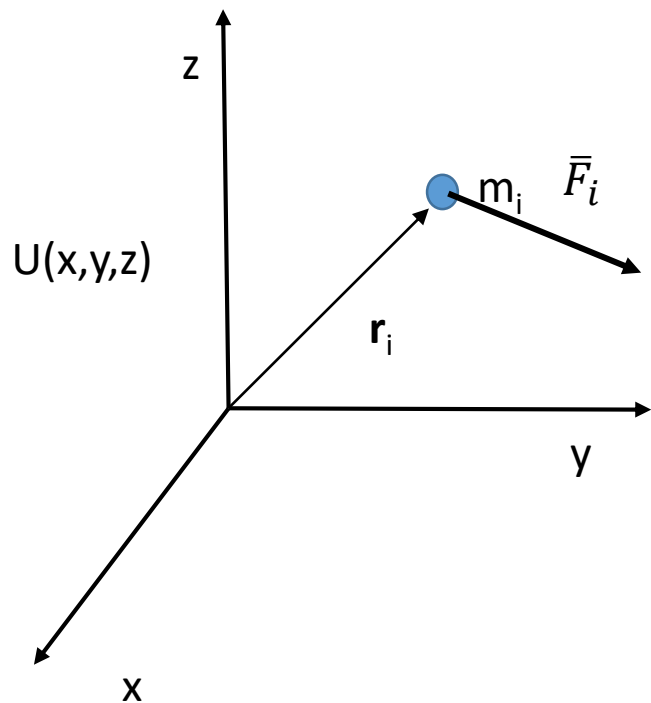
zakładamy, że są równe zero



$$m_1 \ddot{s} + ks + c\dot{s} + \frac{m_2 g s}{L} = P_1(t) - \frac{P_2(t)s}{L}$$

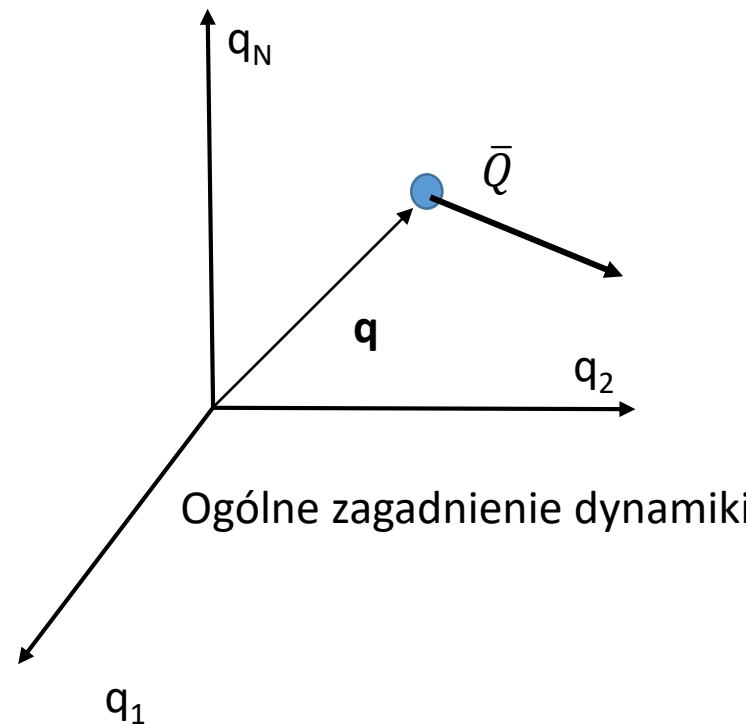
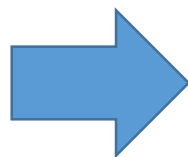
Równanie Lagrange'a w polu sił potencjalnych

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_m} \right) - \frac{\partial E_k}{\partial q_m} = Q_m$$



Pewne pole skalarne

$$U=U(x,y,z)$$



Ogólne zagadnienie dynamiki

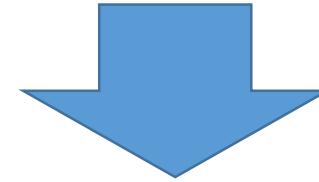
$$\bar{F}_i = grad(U) = - \left[\frac{\partial U}{\partial x_i} \bar{i} + \frac{\partial U}{\partial y_i} \bar{j} + \frac{\partial U}{\partial z_i} \bar{k} \right]$$

we współrzędnych uogólnionych

$$Q_l = -\frac{\partial \pi}{\partial q_l}$$

gdzie

$$\pi = E_p(q_1, q_2, \dots, q_N)$$



Wtedy, jeśli siły są tylko siłami potencjalnymi to:

$$\frac{\partial E_p}{\partial \dot{q}_l} = 0$$

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_m} \right) - \frac{\partial E_k}{\partial q_m} = -\frac{\partial E_p}{\partial q_m} \qquad \frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_m} \right) - \left(\frac{\partial E_k}{\partial q_m} - \frac{\partial E_p}{\partial q_m} \right) = 0$$

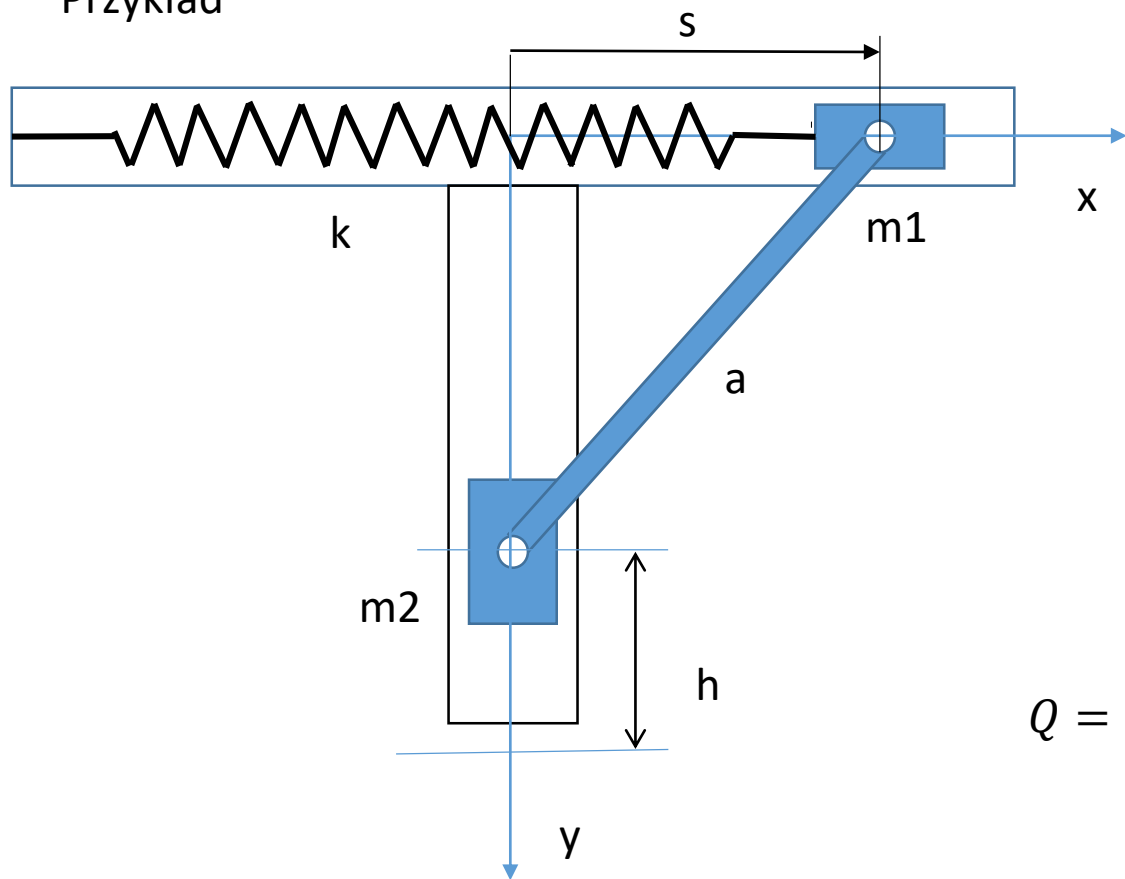
Funkcja Lagrange'a

$$E_k - E_p = L$$

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_m} - \frac{\partial E_p}{\partial \dot{q}_m} \right) - \left(\frac{\partial E_k}{\partial q_m} - \frac{\partial E_p}{\partial q_m} \right) = 0$$

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_m} \right) - \left(\frac{\partial L}{\partial q_m} \right) = 0}$$

Przykład



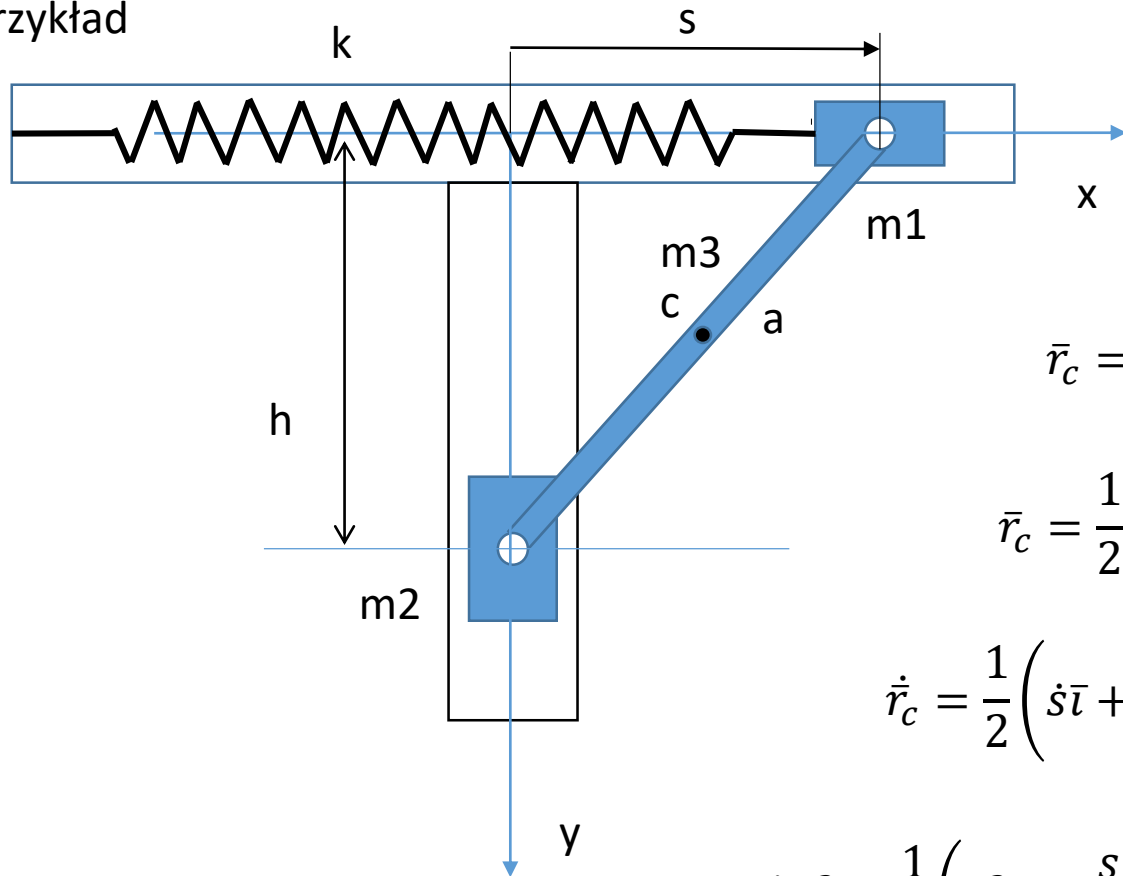
$$E_p = \frac{ks^2}{2} + m_2gh$$

$$h = a - \sqrt{a^2 - s^2}$$

$$Q = -\frac{\partial E_p}{\partial s} = -sk - m_2g \left[-\frac{(-2s)}{2\sqrt{a^2 - s^2}} \right]$$

$$L = E_k - E_p = \frac{1}{2}m_1\dot{s}^2 + \frac{1}{2}m_2\frac{s^2\dot{s}^2}{a^2 - s^2} - \frac{ks^2}{2} - m_2g \left(a - \sqrt{a^2 - s^2} \right)$$

Przykład



Z uwzględnieniem masy pręta

$$\bar{r}_c = \frac{s}{2}\bar{i} + \frac{h}{2}\bar{j} \quad h = \sqrt{a^2 - s^2}$$

$$\bar{r}_c = \frac{1}{2} \left(s\bar{i} + \sqrt{a^2 - s^2}\bar{j} \right)$$

$$\dot{\bar{r}}_c = \frac{1}{2} \left(\dot{s}\bar{i} + \frac{-s\dot{s}}{\sqrt{a^2 - s^2}}\bar{j} \right) = \bar{v}_c$$

$$(\dot{\bar{r}}_c)^2 = \frac{1}{4} \left(\dot{s}^2 + \frac{s^2\dot{s}^2}{a^2 - s^2} \right)$$

$$E_{k3} = \frac{m_3(\dot{\bar{r}}_c)^2}{2} + \frac{I_c(\omega)^2}{2}$$

$$I_c = \frac{m_3 a^2}{12}$$

$$|\dot{\bar{r}}_c| = \omega \frac{a}{2} = v_c$$

$$\omega = \frac{1}{a} \sqrt{\dot{s}^2 + \frac{s^2\dot{s}^2}{a^2 - s^2}}$$

$$E_{k3} = \frac{1}{2} m_3 \frac{1}{4} \left(\dot{s}^2 + \frac{s^2 \dot{s}^2}{a^2 - s^2} \right) + \frac{1}{2} \frac{m_3 a^2}{12} \frac{1}{a^2} \left(\dot{s}^2 + \frac{s^2 \dot{s}^2}{a^2 - s^2} \right) = \frac{m_3}{6} \left(\dot{s}^2 + \frac{s^2 \dot{s}^2}{a^2 - s^2} \right)$$

$$\frac{\partial E_{k3}}{\partial \dot{s}} = \frac{m_3}{6} \left(2\dot{s} + \frac{2s^2 \dot{s}}{a^2 - s^2} \right) = \frac{m_3}{3} \left(\dot{s} + \frac{s^2 \dot{s}}{a^2 - s^2} \right)$$

$$\frac{d}{dt} \left(\frac{\partial E_{k3}}{\partial \dot{s}} \right) = \frac{m_3}{3} \left(\ddot{s} + \frac{(2s \dot{s}^2 + s^2 \ddot{s})(a^2 - s^2) + 2s^3 \dot{s}^2}{(a^2 - s^2)^2} \right) = \frac{m_3}{3} \left(\ddot{s} + \frac{2s \dot{s}^2 a^2 + s^2 \ddot{s} a^2 - 2s^3 \dot{s}^2 - s^4 \ddot{s} + 2s^3 \dot{s}^2}{(a^2 - s^2)^2} \right)$$

$$\frac{\partial E_{k3}}{\partial s} = \frac{m_3}{6} \left(\frac{2s \dot{s}^2 (a^2 - s^2) + s^2 \dot{s}^2 2s}{(a^2 - s^2)^2} \right) = \frac{m_3}{3} \left(\frac{s \dot{s}^2 a^2 - s^3 \dot{s}^2 + s^3 \dot{s}^2}{(a^2 - s^2)^2} \right) = \frac{m_3}{3} \left(\frac{s \dot{s}^2 a^2}{(a^2 - s^2)^2} \right)$$

$$\frac{d}{dt} \left(\frac{\partial E_{k3}}{\partial \dot{s}} \right) - \frac{\partial E_{k3}}{\partial s} = \frac{m_3}{3} \left[\ddot{s} + \frac{2s \dot{s}^2 a^2 + s^2 \ddot{s} a^2 - s^4 \ddot{s}}{(a^2 - s^2)^2} - \frac{s \dot{s}^2 a^2}{(a^2 - s^2)^2} \right] = \frac{m_3}{3} \left[\ddot{s} + \frac{s \dot{s}^2 a^2 + s^2 \ddot{s} a^2 - s^4 \ddot{s}}{(a^2 - s^2)^2} \right]$$

$$E_{p3} = m_3 g \left(-\frac{h}{2} \right) = -\frac{1}{2} m_3 g \sqrt{a^2 - s^2} \quad Q_3 = -\frac{\partial E_p}{\partial s} = \frac{1}{2} m_3 g \frac{-2s}{2\sqrt{a^2 - s^2}} = -\frac{1}{2} m_3 g \frac{s}{\sqrt{a^2 - s^2}}$$