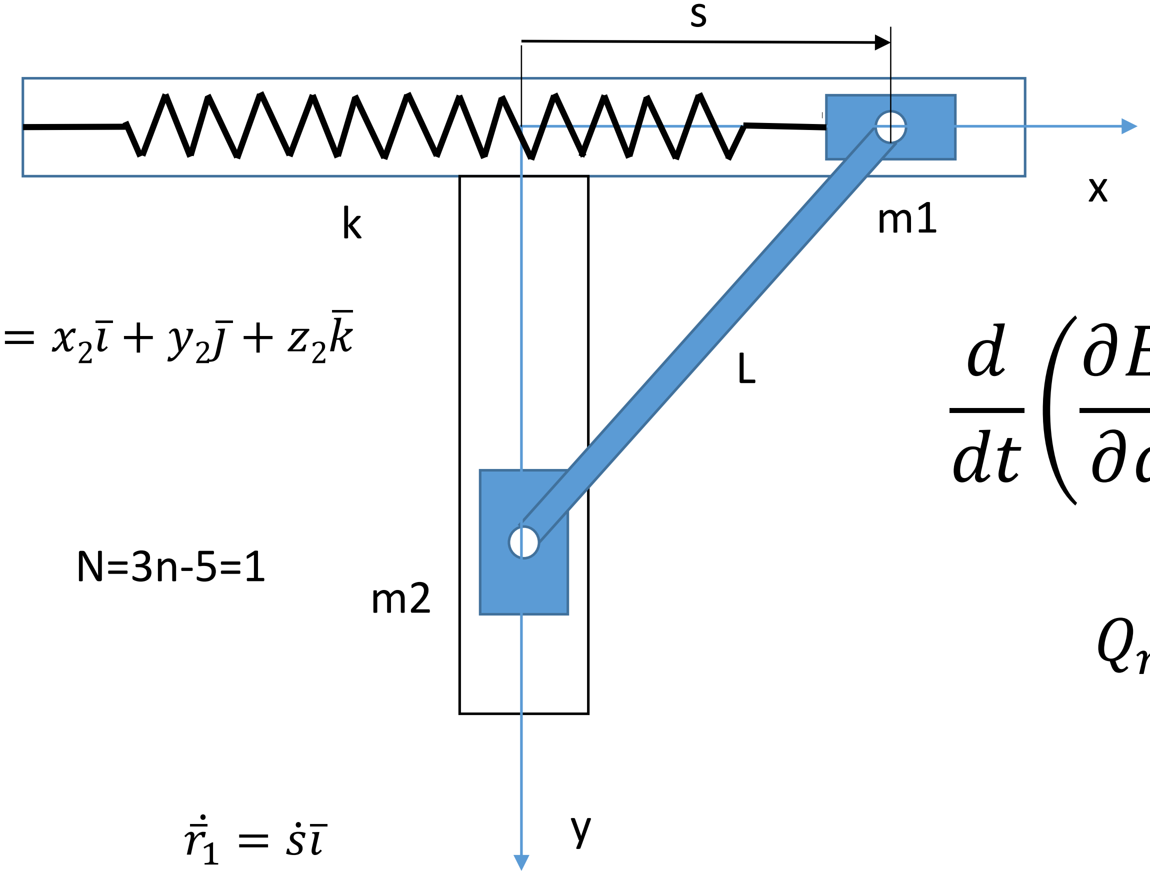
Przykład

Dane: m1,m2,L,k



$$\bar{r}_1 = x_1\bar{\imath} + y_1\bar{\jmath} + z_1\bar{k}, \qquad \bar{r}_2 = x_2\bar{\imath} + y_2\bar{\jmath} + z_2\bar{k}$$

n=2

więzy

z_1=0, z_2=0, y_1=0, x_2=0

$$x_1^2 + y_2^2 = L^2$$

N=3n-5=1

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_m}\right) - \frac{\partial\, E_k}{\partial q_m} = Q_m$$

$$Q_m = \left[\sum_{i=1}^{n} \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_m}\right]$$

$$\bar{r}_1 = s\bar{\imath} \qquad \dot{\bar{r}}_1 = \dot{s}\bar{\imath}$$

$$\bar{r}_2 = y_2\bar{\jmath} = \sqrt{L^2 - s^2}\bar{\jmath} \qquad \dot{\bar{r}}_2 = \frac{-2s\dot{s}}{2\sqrt{L^2 - s^2}}\bar{\jmath}$$

$$E_k = \frac{1}{2}m_1\dot{\bar{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\bar{r}}_2^{\,2} = \frac{1}{2}m_1\dot{s}^2 + \frac{1}{2}m_2\frac{s^2\dot{s}^2}{L^2 - s^2}$$

$$\frac{\partial E_k}{\partial \dot{s}} = m_1\dot{s} + m_2\frac{s^2\dot{s}}{L^2 - s^2}$$

$$\frac{\partial E_k}{\partial s} = \frac{1}{2}m_2\dot{s}^2\left[\frac{2s(L^2 - s^2) - s^2(-2s)}{(L^2 - s^2)^2}\right] = m_2\dot{s}^2\frac{sL^2}{(L^2 - s^2)^2}$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{s}}\right) = m_1\ddot{s} + m_2\left[\frac{\frac{d}{dt}(s^2\dot{s})(L^2-s^2) - (s^2\dot{s})(-2s\dot{s})}{(L^2-s^2)^2}\right] = m_1\ddot{s} + m_2\left[\frac{(2s\dot{s}^2 + s^2\ddot{s})(L^2-s^2) + 2s^3\dot{s}^2}{(L^2-s^2)^2}\right]$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{s}}\right) = m_1\ddot{s} + m_2\frac{s^2\ddot{s}}{(L^2-s^2)} + m_2\left[\frac{2s\dot{s}^2L^2}{(L^2-s^2)^2}\right]$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{s}}\right) - \frac{\partial E_k}{\partial s} = m_1\ddot{s} + m_2\frac{s^2\ddot{s}}{(L^2-s^2)} + m_2\dot{s}^2\frac{2sL^2}{(L^2-s^2)^2} - m_2\dot{s}^2\frac{sL^2}{(L^2-s^2)^2} = m_1\ddot{s} + m_2\frac{s^2\ddot{s}}{(L^2-s^2)} + m_2\frac{\dot{s}^2sL^2}{(L^2-s^2)^2}$$

$$Q = \left[\sum_{i=1}^{n}\bar{F}_i\frac{\partial\bar{r}_i}{\partial s}\right] = \bar{F}_1\frac{\partial\bar{r}_1}{\partial s} + \bar{F}_2\frac{\partial\bar{r}_2}{\partial s}$$

$$\bar{F}_1 = -ks\bar{\imath} + m_1g\bar{\jmath} \qquad \bar{F}_2 = m_2g\bar{\jmath}$$

$$\frac{\partial\bar{r}_1}{\partial s} = \bar{\imath} \qquad \frac{\partial\bar{r}_2}{\partial s} = \frac{-2s}{2\sqrt{L^2-s^2}}\bar{\jmath} = \frac{-s}{\sqrt{L^2-s^2}}\bar{\jmath}$$

$$Q = \bar{\imath}(-ks\bar{\imath} + m_1g\bar{\jmath}) + \frac{-s}{\sqrt{L^2-s^2}}\bar{\jmath}\cdot m_2g\bar{\jmath} = -ks - \frac{m_2gs}{\sqrt{L^2-s^2}}$$

$$m_1\ddot{s} + m_2\frac{s^2\ddot{s}}{(L^2-s^2)} + m_2\frac{\dot{s}^{\,2}sL^2}{(L^2-s^2)^{\,2}} + ks + \frac{m_2 g s}{\sqrt{L^2-s^2}} = 0$$

Równanie jest nieliniowe i skomplikowane

Pierwsze uproszczenie to: małe drgania $s \ll L$ $L^2 - s^2 = L^2$

Elementy wyższego rzędu $s^2\ddot{s}$ i $\dot{s}^{\,2}s$ zakładamy, że są równe zero

$$m_1\ddot{s} + s\left(k + \frac{m_2 g}{L}\right) = 0$$

$$\omega_w = \sqrt{\frac{k + \frac{m_2 g}{L}}{m_1}}$$

oscylator harmoniczny

$$\ddot{x} + \omega_w^2 x = 0$$

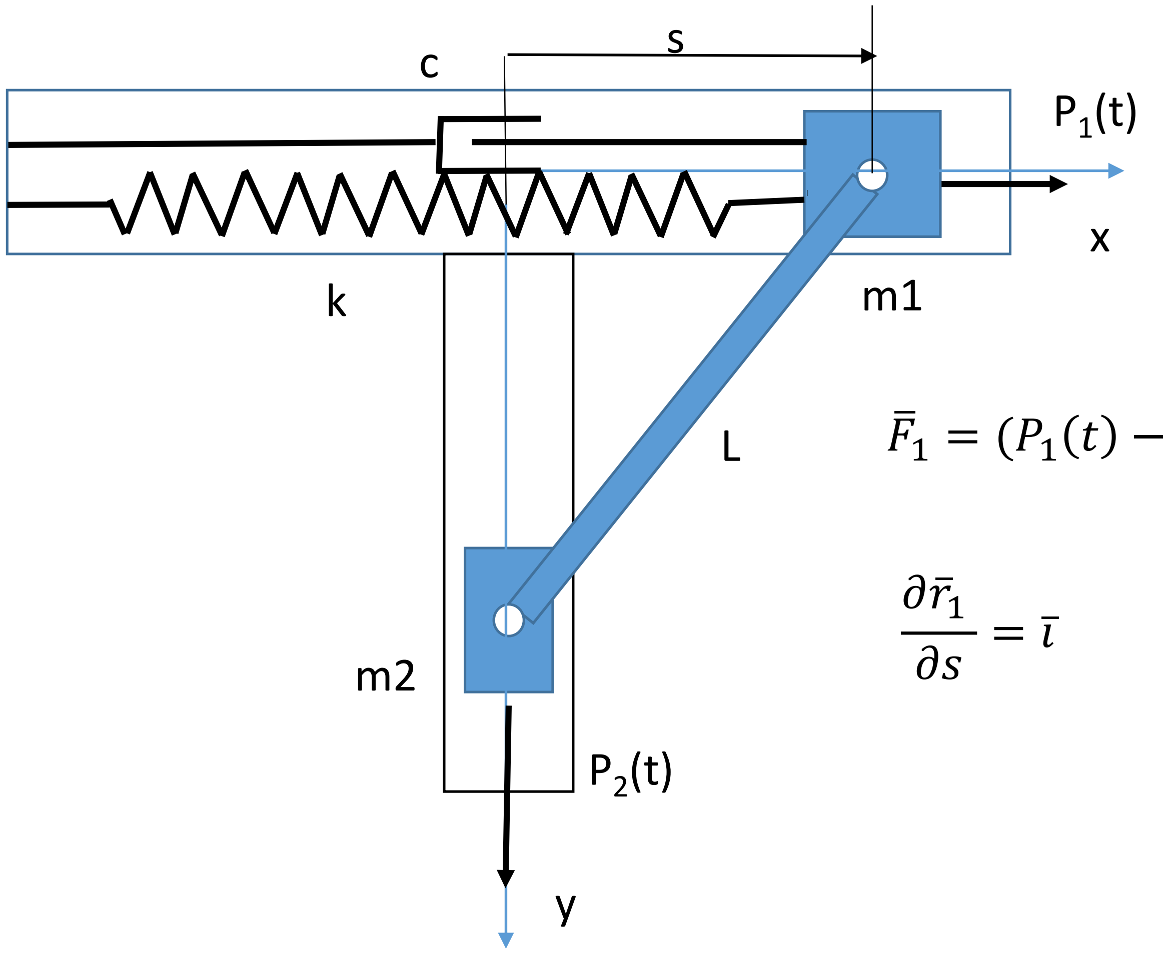
ω_w - częstość drgań własnych

$\omega_w = 2\pi f_w$

częstotliwość

$T = \frac{1}{f_w}$ okres drgań

teraz dodajmy jeszcze tłumienie i siły wymuszające



$$Q = \left[\sum_{i=1}^{n} \bar{F}_i \frac{\partial \bar{r}_i}{\partial s}\right] = \bar{F}_1 \frac{\partial \bar{r}_1}{\partial s} + \bar{F}_2 \frac{\partial \bar{r}_2}{\partial s}$$

$$\bar{F}_1 = (P_1(t) - ks - c\dot{s})\bar{\imath} + m_1 g\bar{\jmath} \qquad \bar{F}_2 = (P_2(t) + m_2 g)\bar{\jmath}$$

$$\frac{\partial \bar{r}_1}{\partial s} = \bar{\imath} \qquad \frac{\partial \bar{r}_2}{\partial s} = \frac{-s}{\sqrt{L^2 - s^2}}\bar{\jmath}$$

$$Q = \bar{\imath}((P_1(t) - ks - c\dot{s})\bar{\imath} + m_1 g\bar{\jmath}) + \frac{-s}{\sqrt{L^2 - s^2}}\bar{\jmath} \cdot (P_2(t) + m_2 g)\bar{\jmath} = P_1(t) - ks - c\dot{s} - \frac{s}{\sqrt{L^2 - s^2}}(P_2(t) + m_2 g)$$

więc równanie będzie miało postać:

$$m_1\ddot{s} + ks + c\dot{s} + m_2\frac{s^2\ddot{s}}{(L^2-s^2)} + m_2\frac{\dot{s}^{\,2}sL^2}{(L^2-s^2)^{\,2}} + \frac{m_2gs}{\sqrt{L^2-s^2}} = P_1(t) - \frac{P_2(t)s}{\sqrt{L^2-s^2}}$$

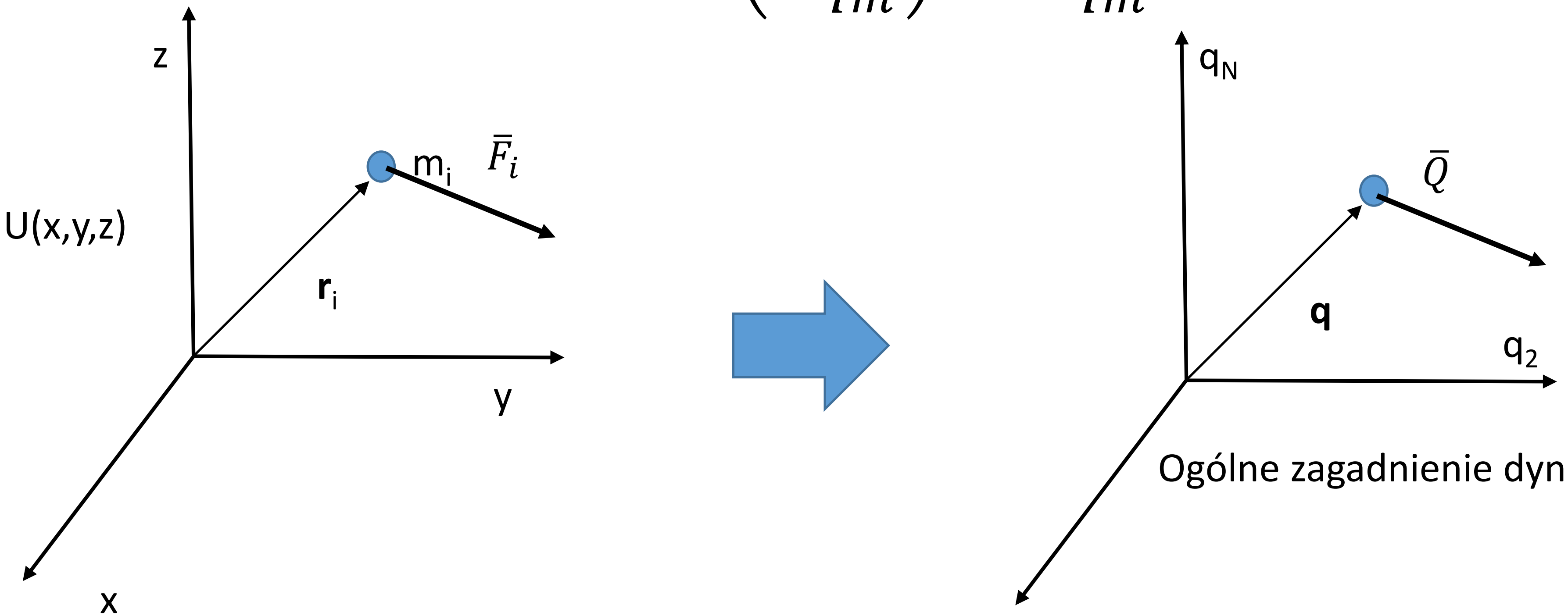
małe drgania → $s \ll L$ → $L^2 - s^2 = L^2$

Elementy wyższego rzędu $s^2\ddot{s}$ i $\dot{s}^{\,2}s$ zakładamy, że są równe zero

$$m_1\ddot{s} + ks + c\dot{s} + \frac{m_2gs}{L} = P_1(t) - \frac{P_2(t)s}{L}$$

Równanie Lagrange'a w polu sił potencjalnych

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_m}\right) - \frac{\partial\, E_k}{\partial q_m} = Q_m$$



Pewne pole skalarne

U=U(x,y,z)

$$\bar{F}_i = grad(U) = -\left[\frac{\partial U}{\partial x_i}\bar{\imath} + \frac{\partial U}{\partial y_i}\bar{\jmath} + \frac{\partial U}{\partial z_i}\bar{k}\right]$$

we współrzędnych uogólnionych $Q_l = -\frac{\partial \pi}{\partial q_l}$ gdzie $\pi = E_p(q_1, q_2, \dots, q_N)$

Wtedy, jeśli siły są tylko siłami potencjalnymi to:

$$\frac{\partial E_p}{\partial \dot{q}_l} = 0$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_m}\right) - \frac{\partial\, E_k}{\partial q_m} = -\frac{\partial E_p}{\partial q_m}$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_m}\right) - \left(\frac{\partial\, E_k}{\partial q_m} + \frac{\partial E_p}{\partial q_m}\right) = 0$$

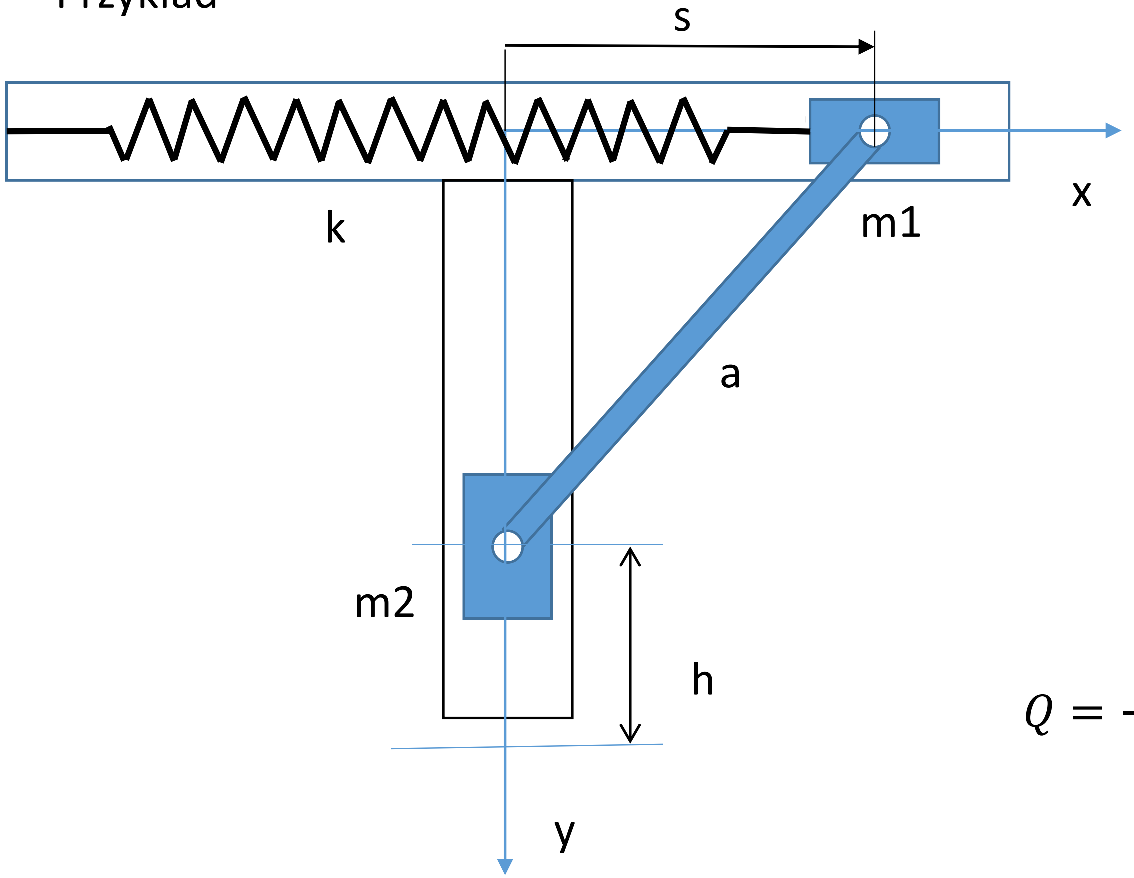
Funkcja Lagrange’a

$$E_k - E_p = L$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_m} - \frac{\partial E_p}{\partial \dot{q}_m}\right) - \left(\frac{\partial\, E_k}{\partial q_m} - \frac{\partial E_p}{\partial q_m}\right) = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) - \left(\frac{\partial\, L}{\partial q_m}\right) = 0}$$

Przykład



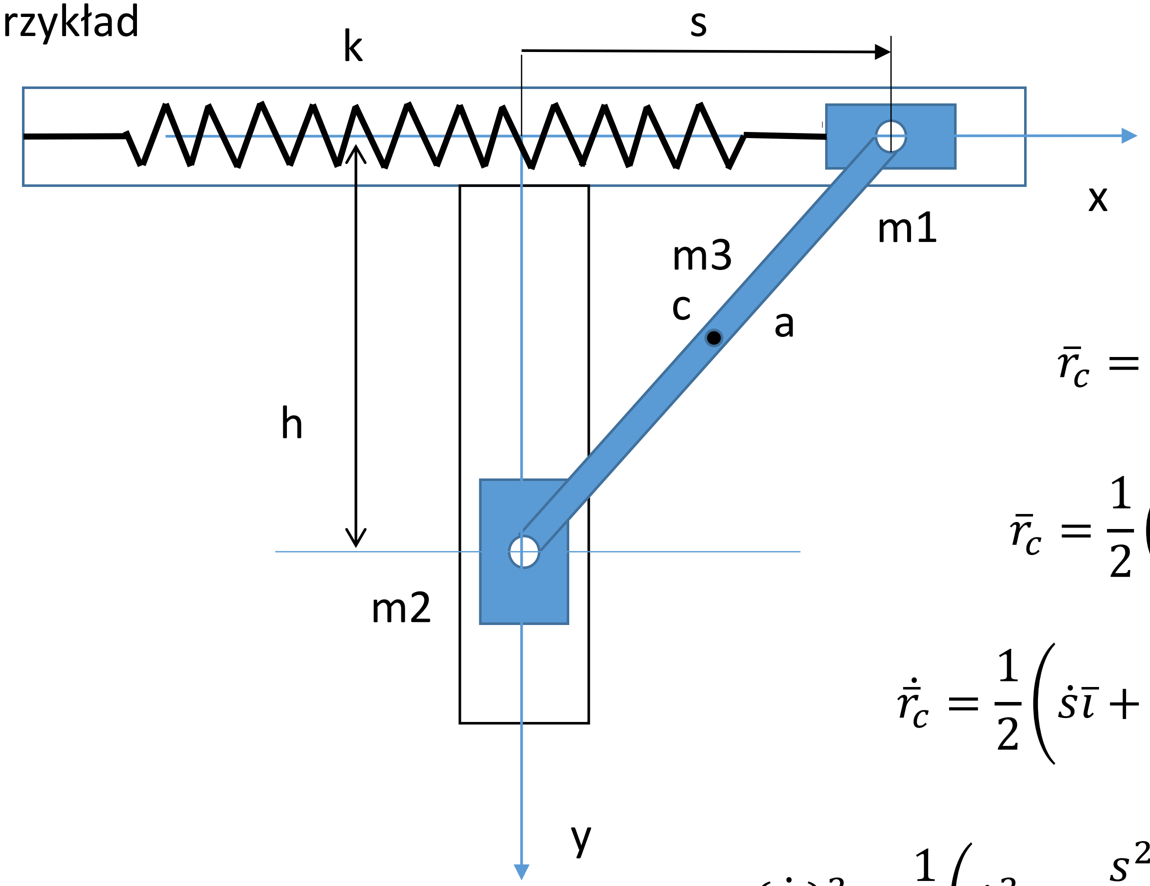
$$E_k = \frac{ks^2}{2} + m_2 gh$$

$$h = a - \sqrt{a^2 - s^2}$$

$$Q = -\frac{\partial E_p}{\partial s} = -sk - m_2 g\left[-\frac{(-2s)}{2\sqrt{a^2 - s^2}}\right]$$

$$L = E_k - E_p = \frac{1}{2}m_1\dot{s}^2 + \frac{1}{2}m_2\frac{s^2\dot{s}^2}{a^2 - s^2} - sk - m_2 g\left[-\frac{(-2s)}{2\sqrt{a^2 - s^2}}\right]$$

Przykład



Z uwzględnieniem masy pręta

$$\bar{r}_c = \frac{s}{2}\bar{\imath} + \frac{h}{2}\bar{\jmath} \qquad h = \sqrt{a^2 - s^2}$$

$$\bar{r}_c = \frac{1}{2}\left(s\bar{\imath} + \sqrt{a^2 - s^2}\bar{\jmath}\right)$$

$$\dot{\bar{r}}_c = \frac{1}{2}\left(\dot{s}\bar{\imath} + \frac{-s\dot{s}}{\sqrt{a^2 - s^2}}\bar{\jmath}\right) = \bar{v}_c$$

$$(\dot{\bar{r}}_c)^2 = \frac{1}{4}\left(\dot{s}^2 + \frac{s^2\dot{s}^2}{a^2 - s^2}\right) \qquad E_{k3} = \frac{m_3(\dot{\bar{r}}_c)^2}{2} + \frac{I_c(\omega)^2}{2}$$

$$I_c = \frac{m_3 a^2}{12} \qquad |\dot{\bar{r}}_c| = \omega\frac{a}{2} = v_c \qquad \omega = \frac{1}{a}\sqrt{\dot{s}^2 + \frac{s^2\dot{s}^2}{a^2 - s^2}}$$

$$E_{k3}=\frac{1}{2}m_3\frac{1}{4}\left(\dot{s}^2+\frac{s^2\dot{s}^2}{a^2-s^2}\right)+\frac{1}{2}\frac{m_3a^2}{12}\frac{1}{a^2}\left(\dot{s}^2+\frac{s^2\dot{s}^2}{a^2-s^2}\right)=\frac{m_3}{6}\left(\dot{s}^2+\frac{s^2\dot{s}^2}{a^2-s^2}\right)$$

$$\frac{\partial E_{k3}}{\partial \dot{s}}=\frac{m_3}{6}\left(2\dot{s}\ +\frac{2s^2\ \dot{s}}{a^2-s^2}\right)=\frac{m_3}{3}\left(\dot{s}\ +\frac{s^2\ \dot{s}}{a^2-s^2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial E_{k3}}{\partial \dot{s}}\right)=\frac{m_3}{3}\left(\ddot{s}\ +\frac{(2s\ \dot{s}^2+s^2\ddot{s})(a^2-s^2)+2s^3\ \dot{s}\ ^2}{(a^2-s^2)\ ^2}\right)=\frac{m_3}{3}\left(\ddot{s}\ +\frac{2s\ \dot{s}^2a^2+s^2\ddot{s}a^2-2s^3\ \dot{s}^2-s^4\ddot{s}+2s^3\ \dot{s}\ ^2}{(a^2-s^2)\ ^2}\right)$$

$$\frac{\partial E_{k3}}{\partial s}=\frac{m_3}{6}\left(\frac{2s\ \dot{s}^2(a^2-s^2)+s^2\dot{s}^2 2s}{(a^2-s^2)^2}\right)=\frac{m_3}{3}\left(\frac{s\ \dot{s}^2a^2-s^3\dot{s}^2+s^3\dot{s}^2}{(a^2-s^2)^2}\right)=\frac{m_3}{3}\left(\frac{s\ \dot{s}^2a^2}{(a^2-s^2)^2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial E_{k3}}{\partial \dot{s}}\right)-\frac{\partial E_{k3}}{\partial s}=\frac{m_3}{3}\left[\ddot{s}\ +\frac{2s\ \dot{s}^2a^2+s^2\ddot{s}a^2-s^4\ddot{s}}{(a^2-s^2)\ ^2}-\frac{s\ \dot{s}^2a^2}{(a^2-s^2)^2}\right]=\frac{m_3}{3}\left[\ddot{s}\ +\frac{s\ \dot{s}^2a^2+s^2\ddot{s}a^2-s^4\ddot{s}}{(a^2-s^2)\ ^2}\right]$$

$$E_{p3}=m_3g\left(-\frac{h}{2}\right)=-\frac{1}{2}m_3g\sqrt{a^2-s^2}\qquad Q_3=-\frac{\partial E_p}{\partial s}=\frac{1}{2}m_3g\frac{-2s}{2\sqrt{a^2-s^2}}=-\frac{1}{2}m_3g\frac{s}{\sqrt{a^2-s^2}}$$